

Spectral statistics for quantum chaos with ray splitting

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Received 1 December 1995; revised manuscript received 12 March 1996; accepted for publication 18 March 1996

Communicated by C.R. Doering

Abstract

We investigate the behavior of ray trajectories and solutions of the wave equation of two dimensional billiard-like systems with ray splitting. By “ray splitting” we mean the phenomenon whereby a ray incident on a sharp boundary leads to two or more rays traveling away from the boundary (e.g. a transmitted ray and a reflected ray). Billiard systems with the same overall shape, but with and without ray splitting boundaries present are examined and compared. It is found that, for the configurations considered, the level spacing distribution and the spectral rigidity for the case without ray splitting are intermediate between Poisson and Gaussian orthogonal ensemble (GOE) statistics, while the behavior with ray splitting is very close to GOE.

PACS: 05.45.+b

Keywords: Resonance spectra; Inhomogeneous media; Semiclassical dynamics

1. Introduction

Quantum chaos focuses on the semi-classical, or short-wavelength, limit of a system described by wave mechanics [1–3]. In this limit the ray equations are just the equations of classical mechanics for the motion of a particle. When the particle motion is chaotic, some very general results have been conjectured to hold, and the behavior of the system should display universal properties. In particular, the energy level distribution is expected to be given by the Gaussian orthogonal ensemble (GOE) if the system obeys an antiunitary symmetry [1,4]. When the phase space consists of a mixture of chaotic orbits and KAM tori, the level distribution is intermediate between the integrable case of Poisson statistics and the GOE case

[5]. At sufficiently short wavelengths, this transition is expected to be universal [5].

In this paper we consider systems for which the ray approximation may be considered to hold everywhere except at an interface where there is an abrupt change of some physical parameter. Examples of this type of system include a quantum particle in a box where the potential has a constant value in one region and jumps discontinuously to a different constant value in a second region [6], an optical system with two regions having different indices of refraction, a bounded elastic medium supporting both shear and compression waves [7,8], and a thin microwave cavity with a sudden change in the cavity height. In this paper we use the thin microwave cavity (Fig. 1) as our model, but the other systems will have very similar behavior. For these systems the usual semi-classical approach is insufficient in that ray trajectories are complicated by

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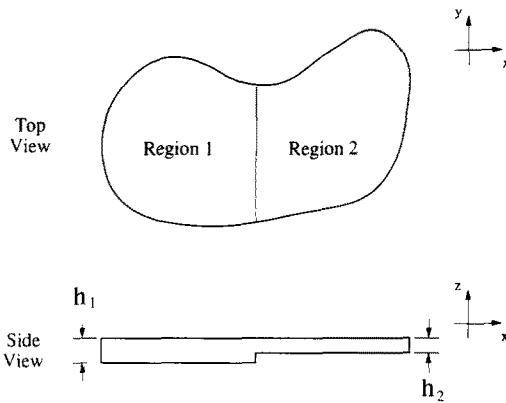


Fig. 1. Microwave cavity with two regions of different thicknesses h_1 and h_2 .

the need to take into account the reflection and transmission which occurs at the interface between the regions.

Systems satisfying a wave equation and having hard boundaries exhibit a type of ray splitting when a ray incident on the boundary excites a so-called “creeping wave” [9]. However, these creeping waves appear at higher order in the semiclassical expansion (higher order in $(kd)^{-1}$, where k is the wavenumber and d is a typical length scale of the system), while ray splitting at an interface occurs at lowest order in $(kd)^{-1}$. In the present work we confine our attention to ray splitting at interfaces.

Previous work [7] has shown that the presence of a ray splitting interface can increase the degree of chaotic behavior of suitably defined ray trajectories in cases where there is a mixed (chaotic and KAM) phase space in the absence of ray splitting. It is natural to ask whether this increase of ray chaos shows up in the quantum spectrum as a shift toward GOE and away from integrable (Poisson) statistics. In fact, Schuetz [10], following Berry [4], shows that the spectral rigidity, $\Delta(l)$, (to be defined below), should have the leading order behavior for the ray splitting case,

$$\Delta(l) = \frac{\alpha}{2\pi^2} \ln(l) + O(1), \quad (1)$$

where α is a constant of order unity which the analysis in Ref. [10] did not pin down. In the nonray splitting case considered by Berry, α is exactly unity, which is the correct leading order behavior for the GOE. The numerical results of the present work indicate that for

systems with ray splitting the spectral statistics are very close to GOE. Thus, when ray splitting is present, GOE statistics can be anticipated in a much wider class of system shapes than has been considered previously.

2. Description of the model

The model system we consider for numerical study is a microwave cavity made of a perfectly conducting material. The cavity is large compared to the typical wavelength in two dimensions, but small enough in the third dimension (which we take as the z direction) so that the electric field is constant in that direction. These waves are described by the z -component of the electric field (all other electric field components are zero), which satisfies the Helmholtz equation in two dimensions,

$$(\nabla^2 + k^2)E = 0, \quad (2)$$

where the wavenumber k is related to the resonant frequency ω of the cavity by $k = \omega/c$, and c is the speed of light. The boundary conditions are that $E = 0$ at the sides of the cavity. Furthermore, we assume that the cavity consists of two regions of different thicknesses h_1 and h_2 in the z direction (Fig. 1), with $kh_{1,2} \gg 1$. The boundary conditions at the interface between the two regions are

$$h_1 E_1 = h_2 E_2, \quad (3)$$

$$\mathbf{n} \cdot \nabla E_1 = \mathbf{n} \cdot \nabla E_2, \quad (4)$$

where $\mathbf{n}$ denotes the unit normal to the boundary.

When constructing the semiclassical (ray) solutions of Eq. (1) one must consider the reflection and transmission of plane waves incident on a boundary such as that depicted in Fig. 1 which is approximated as being locally flat. Using the boundary conditions (3) and (4), the power reflection and transmission coefficients for a ray incident from region 1 to region 2 are

$$R = \frac{(1 - r)^2}{(1 + r)^2}, \quad (5)$$

$$T = \frac{4r}{(1 + r)^2}, \quad (6)$$

where $r = h_1/h_2$. (If the incident wave is in region 2, r should be replaced by $1/r$.) Note that, since Eq. (2)

applies in both regions 1 and 2, with the same value of $k = \omega/c$ in both regions, the angle of incidence and the angle of transmission are the same. Thus, in contrast to a quantum particle in a box with different constant potentials in the two regions, there is no refraction of the transmitted wave. Furthermore, the reflection and transmission coefficients are independent of the angle of incidence, so there is no total internal reflection for this system. This situation may be thought to represent the simplest possible case where ray splitting occurs.

Another situation which has similar characteristics (i.e. absence of refraction at the interface), although with different boundary conditions on the ray splitting surface, occurs for Schrödinger's equation with a potential that is a delta function of strength V_0 on the ray splitting surface. Here the boundary conditions on the wave function become $\psi_1 = \psi_2$ and $\mathbf{n} \cdot \nabla \psi_1 - \mathbf{n} \cdot \nabla \psi_2 = V_0 \psi_{1,2}$. In what follows, with experimental realizability in mind, we shall use the microwave boundary conditions (2) and (3) rather than these quantum boundary conditions.

3. Classical ray mechanics

When there is no ray splitting, the ray approximation to Eq. (2) gives a billiard in two dimensions, with specular reflection at the boundary. For the ray splitting case, we can think about the classical dynamics in the following way: when the particle strikes one of the outer edges it is specularly reflected, but when it strikes an interface between regions with different physical properties it has a probability R of being reflected and a probability T of being transmitted, where R and T are the reflection and transmission coefficients from Eqs. (5), (6). When the ray hits the interface, we randomly choose whether it is reflected or transmitted, according to the probabilities R and T . Between bounces, the particle moves in a straight line. Thus a ray trajectory can be specified by listing the successive points where the ray hits the *side*, and the angle at which it hits each time. (By a *side* we mean *either* an interface between h_1 and h_2 regions *or* a perfectly reflecting billiard wall.) The resulting map preserves phase space area if we use as phase space coordinates the variables (σ, τ) , defined as follows: σ is the distance to the bounce point measured along the boundary from some arbitrary fixed reference point, normalized

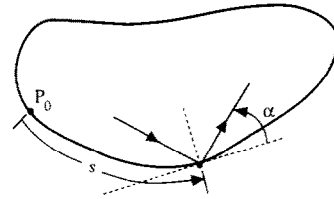


Fig. 2. Coordinates for the ray problem. s is the distance along the perimeter measured from an arbitrary reference point p_0 . $\sigma \equiv s/\text{perimeter}$, and $\tau \equiv \cos \alpha$.

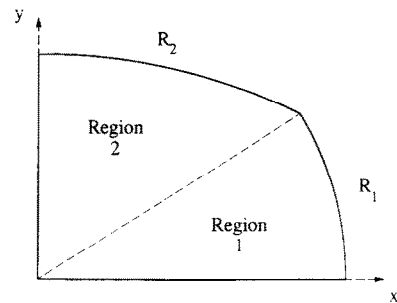


Fig. 3. Cavity used in Sections 3 and 4.

to one, and τ is the cosine of the angle between the momentum vector *after* the bounce and the counter-clockwise tangent to the boundary (see Fig. 2). In the ray splitting case, there are two pieces of phase space corresponding to the two regions, and a ray hitting the interface is plotted in the region of phase space where it ends up after the random choice is made.

Calculations were done for the particular billiard shown in Fig. 3. The right (top) side of the cavity is an arc of a circle of radius R_1 (R_2) whose center lies on the x (y) axis. The circles meet at the point (a, b) . The ray splitting interface is the dashed line from the origin to (a, b) . For the results reported in this paper the parameters are $R_1 = 20$, $R_2 = 6.2$, $(a, b) = (1.6, 1)$, $r = h_1/h_2 = (\sqrt{2} - 1)^2$. For this configuration, if billiards with perfectly reflecting walls were formed for the shapes of each of the subregions, then there is mixed chaotic and KAM behavior, with a significant fraction of the phase space being occupied by KAM tori. This is exhibited in Fig. 4, where Fig. 4a corresponds to region 1 of Fig. 3 and Fig. 4b corresponds to region 2 of Fig. 3. Similarly, with ray splitting removed, Fig. 5a shows that the resulting billiard for the entire region also displays mixed KAM/chaotic behavior. Fig. 5b demonstrates the effect of introducing

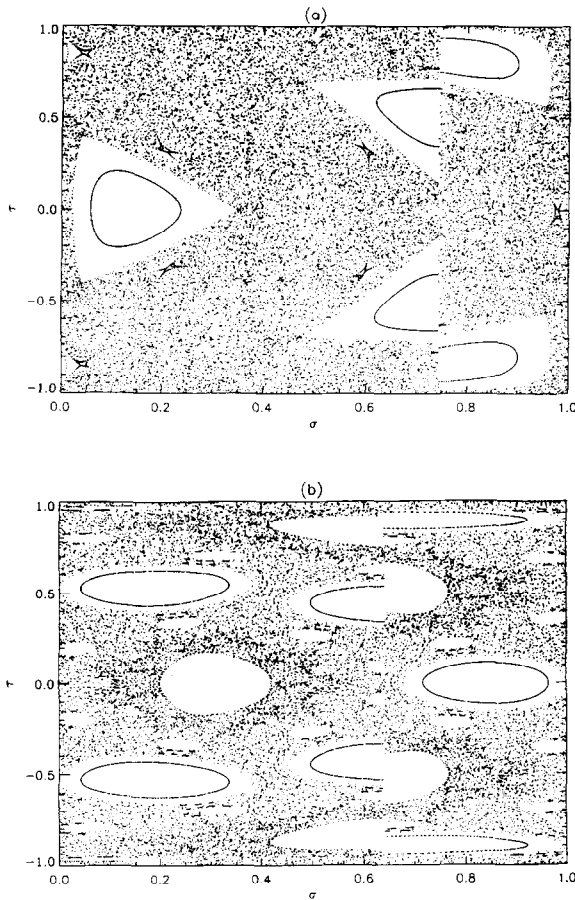


Fig. 4. Classical ray trajectories for (a) region 1 and (b) region 2 of the cavity shown in Fig. 3, considered as separate billiards. In both cases two orbits are plotted, one lying on a KAM torus (1000 iterations of a single initial condition) and one chaotic orbit (20000 iterations of a single initial condition). The discontinuous-looking behavior at certain values of σ is caused by the corners of the billiard, where the tangent to the perimeter jumps discontinuously.

ray splitting. The entire phase space becomes chaotic in the sense that the ray trajectory eventually visits every region of the phase space. Thus all of the KAM tori are destroyed by the ray splitting interface, in contrast to the case that was studied in Ref. [7].

The mechanism for the destruction of KAM tori is quite simple. Suppose a ray starts out on what would be a KAM torus if the ray splitting interface were absent. If it hits the ray splitting interface, it can be transmitted, staying on the original torus, or be reflected into some new orbit. The new orbit could correspond to another KAM torus of the original billiard or to a chaotic orbit. A KAM torus which retains orbits for-

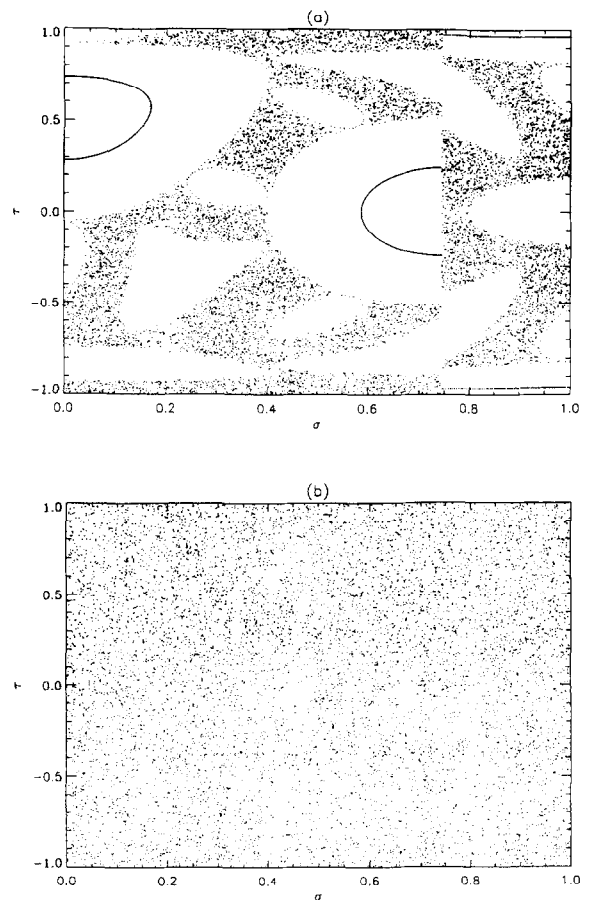


Fig. 5. Classical ray trajectories for the whole cavity of Fig. 3 (a) without ray splitting and (b) with ray splitting. The surface of section is taken to be the boundary of region 2.

ever is thus only possible in the following circumstances: if the torus never intersects the ray splitting surface, or if the portions of the KAM tori defined by the two subregions (assuming reflection) coincide for the part of phase space corresponding to the common boundary (e.g., when the region is a rectangle and the interface is parallel to one of the sides). For the case of Fig. 3 there are no such tori, and therefore all orbits eventually feel the effects of the chaotic fraction of phase space.

In cases where there is a critical angle for total internal reflection, such as the Schrödinger equation with regions of different constant potential, or the elastic medium considered in Ref. [7], a particular periodic orbit for one of the subregions may only hit the interface at angles greater than the critical angle. In such

cases the orbit is not split at the interface. Thus, there may be regions of phase space that retain KAM tori even when ray splitting is present.

4. Chaotic case

The wave properties of the billiards such as the one depicted in Fig. 3 may be discussed in terms of several statistical measures of the spectrum. These statistical measures are defined in terms of the level counting function $N(k^2)$, the number of resonant modes with wavenumber less than a given value k . Define an “unfolded” level counting function $\hat{N}(e)$ by fitting $N(k^2)$ with a quadratic function and then let the unfolded “energies” e_i be defined by

$$e_i = ak_i^2 + bk_i + c, \quad (7)$$

where a , b , and c are the fitting parameters. This gives an unfolded spectrum with unit level density if the resonant levels follow the expected Weyl distribution,

$$N(k^2) \approx \frac{A}{4\pi} k^2 - \frac{P}{4\pi} k, \quad (8)$$

where A is the area and P the perimeter of the region. $N(k^2)$ is called the (smoothed) level counting function. (It is shown in Ref. [11] that this is the correct form of the Weyl formula for a region with Dirichlet boundary conditions on the outer boundary and boundary conditions of the form of Eqs. (3) and (4) on the ray splitting interface.)

Numerical solution of Eq. (2) for the lowest 500 resonant values of k^2 was accomplished using a modified boundary element technique that we have developed for this problem, described in detail in the Appendix.

The probability distribution of level spacings $P(s)$ is defined so that $P(s) ds$ is the probability that s , the normalized separation between neighboring values of k^2 , lies between s and $s+ds$. To obtain s , the separation of neighboring values of k^2 is divided by $dN(k^2)/dk^2$, so that the average value of s is one. For integrable systems a Poisson distribution is expected [12],

$$P(s) = \exp(-s). \quad (9)$$

The Brody distribution [13] is a one-parameter family of distributions given by

$$P_\beta(s) = A_\beta s^\beta \exp(-\alpha s^{1+\beta}), \quad (10)$$

where the normalization is $A_\beta = (1 + \beta)\alpha$,

$$\alpha = \left[\Gamma\left(\frac{2+\beta}{1+\beta}\right) \right]^{1+\beta}, \quad (11)$$

Γ is the gamma function, and β is the Brody parameter. The Brody distribution interpolates between the Poisson distribution, $\beta = 0$, and the Wigner distribution, $\beta = 1$ (which one expects to be valid for a completely chaotic system), so that β can be used as a measure of how close a given distribution is to the two extremes: fully integrable or fully chaotic. We will use the Brody distribution as a convenient measure of the degree of chaoticity of the system.

In the absence of ray splitting we expect to see statistics intermediate between Poisson and GOE [5], because of the mixed phase space evident in Fig. 5a. When ray splitting is present we expect to see purely GOE statistics since the phase space has become completely chaotic (Fig. 5b). The level spacing distribution in the absence of ray splitting is best fit with a Brody parameter of 0.42, clearly showing the effect of having a mixed phase space (Fig. 6a). When ray splitting is introduced (Fig. 6b) the Brody parameter becomes 0.97; very close to the expected value of 1.0 for the GOE spectrum.

Another statistical measure of the resonance spectrum is the spectral rigidity $\Delta(l)$ [14,2]. The spectral rigidity $\Delta(l)$ is defined as the squared deviation of $\hat{N}(e)$ from the best-fitting straight line, integrated over an interval in e of length l , and then averaged over a number of intervals of the same length. The numerical results for the spectral rigidity show a clear shift from intermediate statistics (Fig. 7a) to GOE statistics (Fig. 7b).

Since many physical systems have some amount of ray splitting, and since ray splitting always tends to increase the amount of chaos in the system, our results suggest that GOE-like statistics should be found much more commonly (i.e., for a much less restricted class of shapes) than would be the case in the absence of ray splitting.

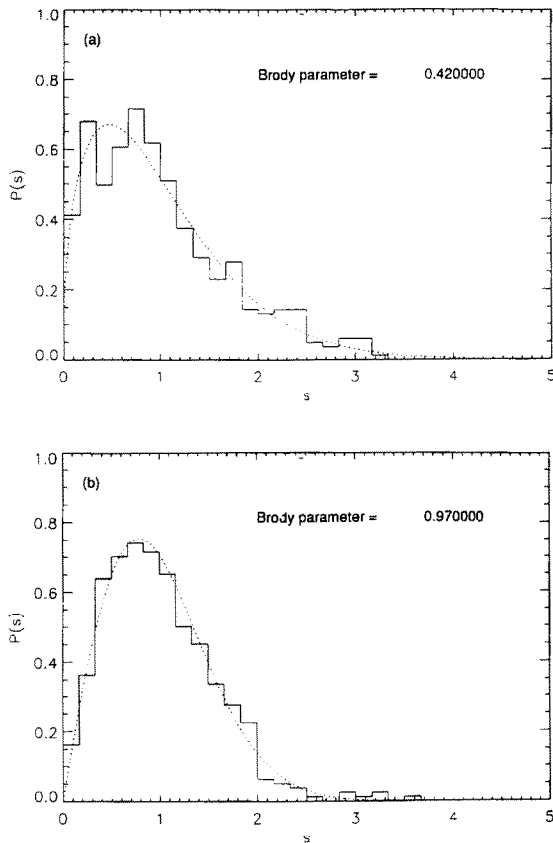


Fig. 6. Histogram of $P(s)$ and the best fitting Brody distribution (dashed line) for the cavity of Fig. 3 (a) without ray splitting and (b) with ray splitting.

5. Conclusions

The model of ray splitting presented in this paper is a particularly straightforward one, in that there is no refraction or critical reflection, and the classical orbits are independent of energy. The model may be realized experimentally by a thin microwave cavity and thus is experimentally testable. (In quantum mechanics somewhat similar behavior is obtained with a delta function potential on the ray splitting surface.)

We have shown how ray splitting causes the destruction of KAM tori in the semi-classical picture. In the full wave solution we have demonstrated a corresponding transition in the spectral statistics. In the absence of ray splitting, the classical phase space is mixed (KAM tori and chaotic regions) and the spectral statistics of the wave solution are intermediate between the statistics expected for an integrable sys-

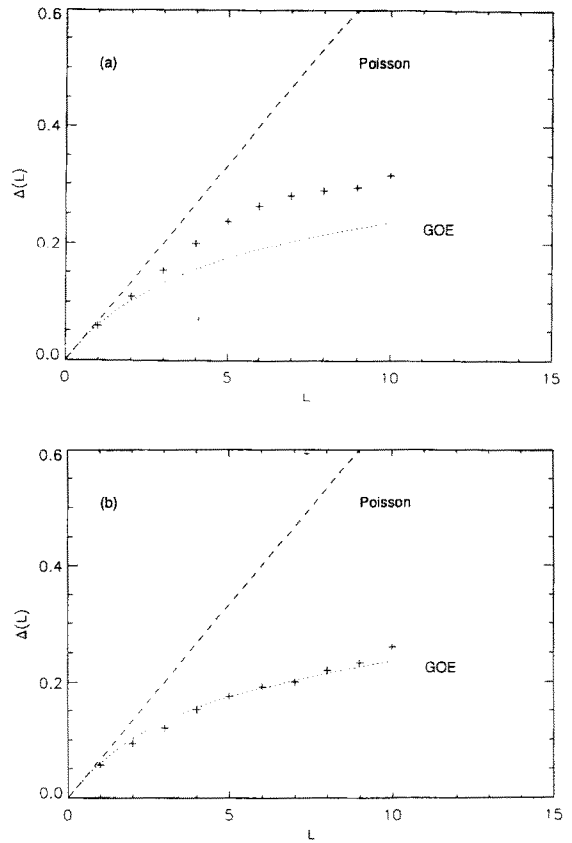


Fig. 7. Spectral rigidity for the cavity of Fig. 3 (a) without ray splitting and (b) with ray splitting.

tem and those expected for a chaotic system. When ray splitting is introduced the KAM tori are destroyed and the spectral statistics become very close to the GOE statistics expected for a classically chaotic system. Thus with ray splitting GOE statistics is to be expected in a much broader class of billiard shapes.

Acknowledgement

This research was supported in part by the Office of Naval Research. R.N.O. was also partially supported by Department of Energy grant DE-FG02-94ER-40854.

Appendix A. Numerical procedure

The solution of the Helmholtz equation (2) in the presence of ray splitting introduces some difficulties which are not present when only a single region is considered. The approach used here is based on the boundary element method, which has been discussed in textbooks on numerical techniques for partial differential equations [15]. Some of these texts discuss the problem of subregions, but we found the standard procedure inadequate for our purposes. In this appendix we outline the modified procedure we used to obtain the results reported in this paper.

An integral representation for the solution to the Helmholtz equation is obtained in the usual way by multiplying Eq. (2) by the appropriate Green function and integrating the result over the region. This is done for each of the two (or more) regions separately. In our case the Green function is the same for both regions, and the reflection comes about solely from the boundary conditions at the interface, Eqs. (3), and (4). For the derivation that follows we allow the two Green functions, G_1 and G_2 , to be different. If E_1 and E_2 represent the solution to the Helmholtz equation in region 1 and 2, respectively, then the boundary integrals are

$$E_1(x) = \int_{S_1} [E_1(y) \nabla_{\hat{n}(y)} G_1(x, y) - G_1(x, y) \nabla_{\hat{n}(y)} E_1(y)] dy, \quad (\text{A.1})$$

$$E_2(x) = \int_{S_2} [E_2(y) \nabla_{\hat{n}(y)} G_2(x, y) - G_2(x, y) \nabla_{\hat{n}(y)} E_2(y)] dy, \quad (\text{A.2})$$

where $S_{1,2}$ are the boundaries to the two regions, including the interface, and $\nabla_{\hat{n}(y)}$ is the normal derivative evaluated at the point y . The “direct” method of solving these equations [15] is to discretize them, impose the boundary conditions (Eqs. (3), and (4) together with $E = 0$ on the exterior boundaries), and then search for values of k for which the determinant of the resulting matrix is zero. In this procedure the unknowns are the values of the normal derivative of $E_{1,2}$ on the exterior boundaries and the value of E_1 and its normal derivative on the interface. Experience with the case of a single region indicates that it is bet-

ter to introduce an auxiliary solution which allows all terms involving the Green function itself to be eliminated, leaving only terms involving its normal derivative. This is preferable because the normal derivative is better behaved than the Green function in the coincidence limit $y \rightarrow x$. In fact, our experience indicates that this “indirect” method is more accurate than the direct method by several orders of magnitude. It seems worthwhile, then, to develop a similar scheme for the case of two regions.

Introduce auxiliary exterior solutions $\tilde{E}_{1,2}$ which are taken to be zero inside the respective region, solve Eq. (2) in the exterior of the region, and have boundary conditions which we are free to choose as we like. These exterior solutions satisfy the same equations (A.2) except that the left-hand side is zero when x is a point in the interior of the respective region. Subtract the equations for $\tilde{E}_{1,2}$ from those for $E_{1,2}$. Defining

$$D_1(y) = E_1(y) - \tilde{E}_1(y), \quad (\text{A.3})$$

$$P_1(y) = \nabla_{\hat{n}(y)} E_1(y) - \nabla_{\hat{n}(y)} \tilde{E}_1(y), \quad (\text{A.4})$$

we find

$$\int_{S_1} [D_1(y) \nabla_{\hat{n}(y)} G_1(x, y) - P_1(y) G_1(x, y)] dy = E_1(x), \quad (\text{A.5})$$

$$\int_{S_2} [D_2(y) \nabla_{\hat{n}(y)} G_2(x, y) - P_2(y) G_2(x, y)] dy = E_2(x). \quad (\text{A.6})$$

Up to this point we have not imposed any boundary conditions. Let us denote the ray splitting interface by S_I , and the remaining boundary of each region, excluding the interface, by $\hat{S}_{1,2}$. For $E_{1,2}$ we have

$$E_1(x) = 0, \quad x \in \hat{S}_1, \quad (\text{A.7})$$

$$E_2(x) = 0, \quad x \in \hat{S}_2, \quad (\text{A.8})$$

and Eqs. (3) and (4) for x on the interface. Choose boundary conditions on $\tilde{E}_{1,2}$ by setting

$$P_1(x) = 0, \quad x \in \hat{S}_1, \quad (\text{A.9})$$

$$P_2(x) = 0, \quad x \in \hat{S}_2, \quad (\text{A.10})$$

$$D_1(x) = 0, \quad x \in S_I, \quad (\text{A.11})$$

$$D_2(x) = 0, \quad x \in S_I. \quad (\text{A.12})$$

A count of the degrees of freedom reveals that we are missing some equations. These are obtained by taking the normal derivative of Eqs. (A.6) with respect to x , for $x \in S_I$. The resulting system of equations is

$$0 = \int_{\hat{S}_1} D_1(y) \nabla_{\hat{n}(y)} G_1(x, y) dy - \int_{S_I} P_1(y) G_1(x, y) dy, \quad x \in \hat{S}_1, \quad (\text{A.13})$$

$$0 = \int_{\hat{S}_2} D_2(y) \nabla_{\hat{n}(y)} G_2(x, y) dy - \int_{S_I} P_2(y) G_2(x, y) dy, \quad x \in \hat{S}_2, \quad (\text{A.14})$$

$$0 = r \int_{\hat{S}_1} D_1(y) \nabla_{\hat{n}(y)} G_1(x, y) dy - \int_{\hat{S}_2} D_2(y) \nabla_{\hat{n}(y)} G_2(x, y) dy \quad (\text{A.15})$$

$$+ \int_{S_I} [P_2(y) G_2(x, y) - r P_1(y) G_1(x, y)] dy, \quad x \in S_I \quad (\text{A.16})$$

$$0 = \int_{\hat{S}_1} D_1(y) \nabla_{\hat{n}(x)} \nabla_{\hat{n}(y)} G_1(x, y) dy - \int_{\hat{S}_2} D_2(y) \nabla_{\hat{n}(x)} \nabla_{\hat{n}(y)} G_2(x, y) dy \quad (\text{A.17})$$

$$+ \int_{S_I} [P_2(y) \nabla_{\hat{n}(x)} G_2(x, y) - P_1(y) \nabla_{\hat{n}(x)} G_1(x, y)] dy, \quad x \in S_I. \quad (\text{A.18})$$

This procedure has introduced the second derivative of the Green function, but because of the particular choice of boundary conditions, Eq. (A.12), the second derivative is never evaluated in the coincidence limit. The price we pay for this is that the terms where the Green function itself needs to be evaluated in the coincidence limit have not been entirely eliminated. However, this procedure does yield a significant improvement in accuracy over the direct method.

The solution follows by discretizing Eqs. (A.18) and searching for the values of the wavenumber, k , for which the determinant of the resulting matrix is zero. The numerical search is facilitated by the fact that the matrix involved contains blocks of zeroes, allowing a block LU decomposition. This procedure yields twice the actual number of eigenvalues, just as in the case of a single homogeneous region. The excess eigenvalues are eliminated using an auxiliary refractive index, in the same way as in that case [16].

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